ANTIHOLOMORPHIC INVOLUTIONS OF SPHERICAL COMPLEX SPACES

DMITRI AKHIEZER AND ANNETT PÜTTMANN

ABSTRACT. Let X be a holomorphically separable irreducible reduced complex space, K a connected compact Lie group acting on X by holomorphic transformations, $\theta : K \to K$ a Weyl involution, and $\mu : X \to X$ an antiholomorphic map satisfying $\mu^2 = \mathrm{Id}$ and $\mu(kx) = \theta(k)\mu(x)$ for $x \in X,\ k \in K$. We show that if $\mathcal{O}(X)$ is a multiplicity free K-module then μ maps every K-orbit onto itself. For a spherical affine homogeneous space $X = G/H$ of the reductive group $G = K^{\mathbb{C}}$ we construct an antiholomorphic map μ with these properties.

1. INTRODUCTION

Let $X = (X, \mathcal{O})$ be a complex space on which a compact Lie group K acts continuously by holomorphic transformations. Then the Fréchet space $\mathcal{O}(X)$ has a natural structure of a K-module. Recall that a K-module W is called *multiplicity free* if any irreducible K-module occurs in W with multiplicity 1 or does not occur at all. A self-map μ of a complex space X is called an *antiholomorphic involution* if μ is antiholomorphic and $\mu^2 = \mathrm{Id}$. For complex manifolds, J.Faraut and E.G.F.Thomas gave an interesting and simple geometric condition which implies that $\mathcal{O}(X)$ is a multiplicity free K-module, see [FT]. Namely, for a complex manifold X the K-action in $\mathcal{O}(X)$ is multiplicity free if

(FT) there exists an antiholomorphic involution $\mu : X \to X$ with the property that, for every $x \in X$, there is an element $k \in K$ such that $\mu(x) = k \cdot x$.

The proof of Theorem 3 in [FT] goes without changes for irreducible reduced complex spaces. It should be noted that the setting in [FT] is more general. Namely, the authors consider any, not necessarily compact, group of holomorphic transformations of X and study invariant Hilbert subspaces of $\mathcal{O}(X)$. We will give a simplified proof of their result in our context, see Proposition 1. Our main purpose, however, is to prove the converse theorem for a special class of manifolds, namely, for Stein (or, equivalently, affine algebraic) homogeneous spaces of complex reductive groups.

Let G be a connected reductive complex algebraic group and $K \subset G$ a maximal compact subgroup. We prove that an affine homogeneous space $X = G/H$ is spherical (or, equivalently, $\mathcal{O}(X)$ is a multiplicity free K-module) if and only if the K-action on X satisfies (FT), see Theorem 4. Recall that a diffeomorphism μ of a manifold X with a K-action is said to be *θ-equivariant*, if θ is an automorphism of K and $\mu(kx) = \theta(k)\mu(x)$ for all $x \in X,\ k \in K$. For $X = G/H$ spherical we can say more about μ in (FT). Namely, again by Theorem 4, μ can be chosen θ-equivariant, where θ is a Weyl involution of K.

Research supported by SFB/TR12 "Symmetrien und Universalität in mesoskopischen Systemen" of the Deutsche Forschungsgemeinschaft. First author supported in part by the Russian Foundation for Basic Research, Grant 04-01-00647.

In order to prove Theorem 4, we consider θ-equivariant antiholomorphic involutions in a more general context. Namely, let X be a holomorphically separable irreducible reduced complex space, K a connected compact Lie group of holomorphic transformations of X, and μ an antiholomorphic θ-equivariant involution of X. Then our Theorem 1 asserts that $\mathcal{O}(X)$ is multiplicity free if and only if $\mu(x) \in Kx$ for all $x \in X$, i.e., if X has property (FT) with respect to μ.

Another important ingredient in the proof of Theorem 4 is the construction of two commuting involutions of G, a Weyl involution and a Cartan involution, which both preserve a given reductive spherical subgroup $H \subset G$, see Theorem 3. The proof is based on the results of [AV] and, therefore, on the classification of spherical subgroups.

At the end of our paper we give an example of an affine homogeneous space without θ-equivariant antiholomorphic involutions, see Proposition 4.

2. Fourier series of Harish-Chandra

Harish-Chandra carried over the classical Fourier series to the representation theory of compact Lie groups in Fréchet spaces, see [H-C]. In this paper, we will need only the representations in $\mathcal{O}(X)$, where X is a complex space. We recall the result of Harish-Chandra in this setting. The details can be found in [H-C], see also [A], Ch. 5.

Let K be a compact Lie group, $\hat{K}$ its unitary dual, dk the normalized Haar measure on K. For $\delta \in \hat{K}$ let χ_δ denote the character of δ multiplied by the dimension of δ. Suppose that K acts by holomorphic transformations on a complex space X. Then we have a continuous representation of K in $\mathcal{C}(X)$ and in $\mathcal{O}(X)$. We will assume that X is reduced and irreducible, so the representation is given by $k \cdot f(x) = f(k^{-1}x)$, where $k \in K, x \in X$.

Define an operator family $\{E_\delta\}_{\delta \in \hat{K}}$ in $\mathcal{O}(X)$ by

$$E_\delta f(x) = \int_K \overline{\chi_\delta(k)} \cdot f(k^{-1}x) \cdot dk.$$

From orthogonality relations for characters it follows that all E_δ commute with the representation of K. Furthermore, $\{E_\delta\}_{\delta \in \hat{K}}$ is a family of projection operators, i.e. $E_\delta^2 = E_\delta$ and $E_\delta E_\epsilon = 0$ if $\delta \neq \epsilon$. Let $\mathcal{O}_\delta(X) = E_\delta \mathcal{O}(X)$. Then $\mathcal{O}_\delta(X) = \mathrm{Ker}(E_\delta - \mathrm{Id})$, so $\mathcal{O}_\delta(X)$ is a closed subspace. Again from orthogonality relations it follows that $\mathcal{O}_\delta(X)$ is the *isotypic component of type* δ, i.e., $\mathcal{O}_\delta(X)$ consists of all those vectors in $\mathcal{O}(X)$ whose K-orbit is contained in a finite-dimensional K-submodule where the representation is some multiple of δ.

Harish-Chandra proved that each $f \in \mathcal{O}(X)$ can be uniquely written in the form

$$f = \sum_{\delta \in \hat{K}} f_\delta,$$

where $f_\delta = E_\delta(f) \in \mathcal{O}_\delta(X)$ and the convergence is absolute and uniform on compact subsets in X.

Assume now that L is another compact Lie group acting by holomorphic transformations of another complex space Y subject to our assumptions. We will use similar notation for L, in particular, θ_ϵ will denote the character of $\epsilon \in \hat{L}$ multiplied by the dimension of ϵ. For the representation of $K \times L$ in $\mathcal{O}(X \times Y)$ defined by

$$(k, l) \cdot f(x, y) = f(k^{-1}x, l^{-1}y),\ x \in X, y \in Y, k \in K, l \in L,$$

the type of an isotypic component is determined by a pair $\delta \in \hat{K}, \epsilon \in \hat{L}$. The corresponding isotypic component will be denoted $\mathcal{O}_{\delta,\epsilon}(X \times Y)$. Of course, the tensor product $\mathcal{O}_\delta(X) \otimes \mathcal{O}_\epsilon(Y)$ is contained in $\mathcal{O}_{\delta,\epsilon}(X \times Y)$. We will need the following lemma.

Lemma 1. *If* $\dim \mathcal{O}_\delta(X) < \infty$ *for some* $\delta \in \hat{K}$ *then* $\mathcal{O}_{\delta,\epsilon}(X \times Y) = \mathcal{O}_\delta(X) \otimes \mathcal{O}_\epsilon(Y)$ *for all* $\epsilon \in \hat{L}$.

Proof. Let $f \in \mathcal{O}_{\delta,\epsilon}(X \times Y)$, then

$$\begin{aligned} f(x,y) &= \int_{K \times L} \overline{\chi_\delta(k)\theta_\epsilon(l)} \cdot f(k^{-1}x, l^{-1}y) \cdot dk dl = \\ &= \int_L \overline{\theta_\epsilon(l)} \Big(\int_K \overline{\chi_\delta(k)} \cdot f(k^{-1}x, l^{-1}y) \cdot dk \Big) \, dl \end{aligned}$$

by Fubini theorem. The function

$$x \mapsto \int_K \overline{\chi_\delta(k)} \cdot f(k^{-1}x, y) \cdot dk$$

is in $\mathcal{O}_\delta(X)$ for all $y \in Y$. Let $\{\varphi_i\}_{i=1,\dots,N}$ be a basis of $\mathcal{O}_\delta(X)$. Then

$$\int_K \overline{\chi_\delta(k)} \cdot f(k^{-1}x, y) \cdot dk = \sum_{i=1}^{N} c_i(y)\varphi_i(x)$$

with some $c_i \in \mathcal{O}(Y)$. Replace in this equality y by $l^{-1}y$, multiply it by $\overline{\theta_\epsilon(l)}$ and integrate over L against the Haar measure dl. Then we get

$$f(x,y) = \sum_{i=1}^{N} \varphi_i(x)\psi_i(y),$$

where

$$\psi_i(y) = \int_L \overline{\theta_\epsilon(l)} \cdot c_i(l^{-1}y) \cdot dl \ \in \mathcal{O}_\epsilon(Y).$$

□

3. K-action and complex conjugation

As in the previous section, X is an irreducible reduced complex space and K is a compact group acting on X by holomorphic transformations.

Lemma 2. *Let* $W \subset \mathcal{O}(X)$ *be a finite-dimensional* K*-submodule. Introduce a* K*-invariant Hermitian inner product and choose a unitary basis* $\{f_1, \dots, f_N\}$ *in* W*. The function* $F := \sum_{j=1}^N f_j \overline{f_j}$ *is* K*-invariant. Furthermore,* F *does not depend on the choice of basis.*

Proof. Let $\{g_1, \dots, g_N\}$ be another unitary basis of W. There is a unitary transformation $A : W \to W$ such that $A(f_j) = g_j = \sum_{l=1}^N a_{lj} f_l$. We have

$$\sum_{j=1}^{N} g_j \overline{g_j} = \sum_{j=1}^{N} \sum_{l,l'=1}^{N} a_{lj} \overline{a_{l'j}} f_l \overline{f_{l'}} = \sum_{l,l'=1}^{N} \delta_{ll'} f_l \overline{f_{l'}} = F.$$

Now, $k \cdot F = \sum_{j=1}^N (k \cdot f_j)\overline{(k \cdot f_j)}$ for $k \in K$. But $\{k \cdot f_1, \dots, k \cdot f_N\}$ is another unitary basis of W. Since F does not depend on the choice of basis, it follows that $k \cdot F = F$ for any $k \in K$. □

Lemma 3. *If $W \subset \mathcal{O}(X)$ is a finite-dimensional K-submodule, then $\overline{W} \subset \overline{\mathcal{O}(X)}$ is also a K-submodule, which is isomorphic to the dual module W^*.*

Proof. Let (f, g) be a K-invariant Hermitian product on W. For $f \in W,\ \phi \in \overline{W}$ we have the bilinear pairing

$$\langle f, \phi \rangle = (f, \overline{\phi}),$$

which is obviously K-invariant and non-degenerate. This shows that $\overline{W}$ is isomorphic to W^*. □

Let $\mu : X \to X$ be an antiholomorphic involution. Then, by definition, the function $\mu f(x) = f(\mu x)$ is antiholomorphic for any $f \in \mathcal{O}(X)$. We want to give a simple proof of the theorem of J.Faraut and E.G.F.Thomas in our setting.

Proposition 1. *If the K-action on X satisfies (FT) then $\mathcal{O}(X)$ is a multiplicity free K-module.*

Proof. Assume the contrary. Let $W, W' \subset \mathcal{O}(X)$ be two irreducible isomorphic K-submodules, such that $W \neq W'$. Define $f_1, \dots, f_N \in W$ as in Lemma 2. Fix a K-equivariant isomorphism $\phi : W \to W'$ and let $f_i' = \phi(f_i)$. By Lemma 2 the function $F = \sum f_i \overline{f_i}$ is K-invariant. The same proof shows that the function $G = \sum f_i \overline{f_i'}$ is also K-invariant. By (FT) we have $\mu F = F$ and $\mu G = G$. Since the multiplication map $\mathcal{O}(X) \otimes \overline{\mathcal{O}(X)} \to \mathcal{O}(X) \cdot \overline{\mathcal{O}(X)}$ is an isomorphism of vector spaces, it follows that

$$\sum_i \overline{\mu f_i} \otimes \mu f_i = \sum_i f_i \otimes \overline{f_i}$$

and

$$\sum_i \overline{\mu f_i'} \otimes \mu f_i = \sum_i f_i \otimes \overline{f_i'}.$$

Therefore the linear span of $\overline{\mu f_1}, \dots, \overline{\mu f_N}$ coincides with the linear span of $f_1, \dots, f_N$ and with the linear span of $\overline{\mu f_1'}, \dots, \overline{\mu f_N'}$. Thus $\overline{\mu W} = \overline{\mu W'}$ and $W = W'$, contradictory to our assumption. □

From now on the compact group K is assumed connected. An involutive automorphism $\theta : K \to K$ is called a *Weyl involution* if there exists a maximal torus $T \subset K$ such that $\theta(t) = t^{-1}$ for $t \in T$. It is known that Weyl involutions exist and that they are all conjugated by inner automorphisms. If θ is a Weyl involution and ρ a linear representation of K then $\rho \circ \theta$ is the dual representation. Recall that an antiholomorphic involution $\mu : X \to X$ is called θ-equivariant if $\mu(kx) = \theta(k)\mu(x)$ for all $x \in X,\ k \in K$.

Lemma 4. *Let θ be a Weyl involution of K and $\mu : X \to X$ a θ-equivariant antiholomorphic involution of X. If $W \subset \mathcal{O}(X)$ is a finite-dimensional K-submodule then μW is also a K-submodule. Furthermore, $\overline{W}$ and μW are isomorphic K-modules.*

Proof. Introduce a K-invariant Hermitian inner product and choose a unitary basis $\{f_1, \dots, f_N\}$ in W. Denote the representation in W by ρ. The condition $\mu(kx) = \theta(k)\mu(x)$ implies that

$$k \cdot \mu f(x) = \mu f(k^{-1}x) = f(\mu(k^{-1}x)) = f(\theta(k)^{-1}\mu(x)) = \theta(k) f(\mu x) = \mu\theta(k) f(x).$$

Hence μW is indeed a K-submodule with the representation $\rho \circ \theta$. Since θ is a Weyl involution, this representation is dual to ρ. But the representation in $\overline{W}$ is also dual to ρ by Lemma 3, and our assertion follows. $\square$

Lemma 5. *Keep the notation of Lemma 4 and assume in addition that W is irreducible and $\mu W = \overline{W}$. Then for a K-invariant Hermitian inner product on W one has*

$$(\overline{\mu f_1}, \overline{\mu f_2}) = (f_1, f_2),$$

where $f_1, f_2 \in W$.

Proof. The new Hermitian inner product $\{f_1, f_2\} := (\overline{\mu f_1}, \overline{\mu f_2})$ on W is also K-invariant. Since W is an irreducible K-module, it follows that $\{f_1, f_2\} = c(f_1, f_2)$, where $c > 0$. But then

$$\{\overline{\mu f_1}, \overline{\mu f_2}\} = c(\overline{\mu f_1}, \overline{\mu f_2})$$

and, on the other hand,

$$\{\overline{\mu f_1}, \overline{\mu f_2}\} = (f_1, f_2)$$

because μ is an involution. Thus

$$c(\overline{\mu f_1}, \overline{\mu f_2}) = (f_1, f_2) = c^{-1}(\overline{\mu f_1}, \overline{\mu f_2}),$$

hence $c^2 = 1$ and $c = 1$. $\square$

4. Holomorphically separable spaces

Since we assume that K is connected, the irreducible representations of K are determined by their highest weights. We denote by W_λ an irreducible K-module with highest weight λ and write $\mathcal{O}_\lambda(X)$ instead of $\mathcal{O}_\delta(X)$, where $\delta \in \hat{K}$ and $\lambda = \lambda(\delta)$ is the highest weight of δ. Those highest weights λ, for which W_λ occurs in our K-module $\mathcal{O}(X)$, form an additive semigroup, to be denoted by $\Lambda(X)$. In other words, $\Lambda(X)$ is the set of highest weights such that $\mathcal{O}_\lambda(X) \neq \{0\}$. The subspace of fixed vectors of a K-module W is denoted by W^K. We remark that if A is an algebra on which K acts as a group of automorphisms then A^K is a subalgebra of A.

Theorem 1. *Let X be a holomorphically separable irreducible reduced complex space, K a connected compact Lie group acting on X by holomorphic transformations, $\theta : K \to K$ a Weyl involution and $\mu : X \to X$ a θ-equivariant antiholomorphic involution of X. Then $\mathcal{O}(X)$ is multiplicity free if and only if $\mu(x) \in Kx$ for all $x \in X$.*

Proof. If $\mu(x) \in Kx$ for all $x \in X$, then (FT) guarantees that the K-action on $\mathcal{O}(X)$ is multiplicity free, see Introduction and Proposition 1.

We now prove the converse. Let $\mathcal{A}(X) = \mathcal{O}(X) \cdot \overline{\mathcal{O}(X)}$. Since X is holomorphically separable, the algebra $\mathcal{A}(X)$ separates points of X. By Stone-Weierstrass theorem $\mathcal{A}(X)$ is dense in the algebra $\mathcal{C}(X)$ of continuous functions on X. The standard averaging argument shows that $\mathcal{A}(X)^K$ is dense in $\mathcal{C}(X)^K$. Now, if Kx and Ky are two different K-orbits in X then there is a K-invariant continuous function $f \in \mathcal{C}(X)$ which seperates these orbits. Since this function can be approximated by K-invariant functions from $\mathcal{A}(X)$, it follows that $\mathcal{A}(K)^K$ separates K-orbits. Let $\lambda \in \Lambda(X)$ be a highest weight which occurs in the decomposition of the K-algebra $\mathcal{O}(X)$. Since $\mathcal{O}(X)$ is multiplicity free, the isotypic component $\mathcal{O}_\lambda(X)$ is irreducible. We can identify this isotypic component with W_λ, and so we

write $W_\lambda = \mathcal{O}_\lambda(X)$. Now apply Lemma 2 to construct a K-invariant function in $W_\lambda \cdot \overline{W_\lambda}$. Call this function F_λ. We claim that the family $\{F_\lambda\}_{\lambda\in\Lambda(X)}$ also separates K-orbits in X.

To prove the claim it is enough to present each $F \in \mathcal{A}(X)^K$ as the sum of a series

$$F = \sum_{\lambda\in\Lambda(X)} c_\lambda F_\lambda \ ,$$

where the convergence is absolute and uniform on compact subsets in X. In order to prove this decomposition, consider the complex space $\overline{X}$ with the conjugate complex structure. There is a natural K-action on $\overline{X}$, and so we obtain an action of $K\times K$ on $X\times\overline{X}$. Since $\mathcal{O}(\overline{X}) = \overline{\mathcal{O}(X)}$, the isotypic components of the K-module $\mathcal{O}(\overline{X})$ are just the submodules $\overline{W_\lambda}$. By Lemma 1 the isotypic components of the $(K\times K)$-module $\mathcal{O}(X\times\overline{X})$ are the tensor products $W_\lambda \otimes \overline{W_{\lambda'}}$.

For any $F \in \mathcal{O}(X\times\overline{X})$ the theorem of Harish-Chandra yields the decomposition

$$F = \sum F_{\lambda\lambda'} \ \text{ with } F_{\lambda\lambda'} \in W_\lambda \otimes \overline{W_{\lambda'}},$$

where the convergence is absolute and uniform on compact subsets in $X\times\overline{X}$. In particular, if $F \in (\mathcal{O}(X)\otimes\overline{\mathcal{O}(X)})^K$ then all summands are K-invariant. But $\overline{W_\lambda}$ is dual to W_λ by Lemma 3, hence $F_{\lambda\lambda'} = 0$ for $\lambda' \neq \lambda$ by Schur lemma. The remaining summands $F_{\lambda\lambda}$ are K-invariant elements in $W_\lambda \otimes \overline{W_\lambda}$. But the space $(W_\lambda \otimes \overline{W_\lambda})^K$ is one-dimensional, again by Schur lemma. Therefore, restricting $F_{\lambda\lambda}$ to the diagonal in $X\times\overline{X}$, we get the functions proportional to the F_λ's defined above.

Now, because $\mathcal{O}(X)$ is multiplicity free, it follows from Lemma 4 that $\overline{W_\lambda} = \mu W_\lambda$. Furthermore, Lemma 5 shows that the composition of μ with complex conjugation preserves a K-invariant Hermitian product on W_λ. Therefore $\mu F_\lambda = F_\lambda$ by Lemma 2. Since the family of functions F_λ separates K-orbits, μ must preserve each of them or, equivalently, $\mu x \in Kx$ for all $x \in X$. □

Remark For the torus $T = (S^1)^m$ the Weyl involution is given by $\theta(t) = t^{-1}$. Suppose that T acts on $\mathbb{P}_n$ by $t\cdot(z_0 : \ldots : z_n) = (\chi_0(t)z_0 : \ldots : \chi_n(t)z_n)$ with some characters $\chi_i : T \to S^1,\ i = 0,\ldots,n$, and $\mu : \mathbb{P}_n \to \mathbb{P}_n$ is given by $\mu(z_0 : \ldots : z_n) = (\overline{z_0} : \ldots : \overline{z_n})$. Then μ is obviously θ-equivariant. However, if $m < n$ then μ cannot map each T-orbit onto itself. This shows that holomorphic separability of X in Theorem 1 is essential.

Let $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$ its complexification. An irreducible reduced complex space X is called *spherical* under the action of a compact connected Lie group K, if X is normal and there exists a point $x \in X$ such that the tangent space T_xX is generated by the elements of a Borel subalgebra of $\mathfrak{g}$, see[AH].

Theorem 2. *Let X be a normal Stein space, K a connected compact Lie group acting on X by holomorphic transformations, $\theta : K \to K$ a Weyl involution and $\mu : X \to X$ a θ-equivariant antiholomorphic ivolution of X. Then X is spherical if and only if $\mu(x) \in Kx$ for all $x \in X$.*

Proof. It is known that a normal Stein space X is spherical if and only if $\mathcal{O}(X)$ is a multiplicity free K-module [AH]. The result follows from Theorem 1. □

5. Weyl involution and Cartan involution

Throughout this section, except Theorem 4, the word *involution* means an involutive automorphism of a group. This notion will be used for complex algebraic groups and for Lie groups. Let G be a connected reductive algebraic group over $\mathbb{C}$ and let K be a connected compact Lie group. So far we considered Weyl involutions of K, but they can be also defined for G. Namely, an involution $\theta : G \to G$ is called a *Weyl involution* if there exists a maximal algebraic torus $T \subset G$ such that $\theta(t) = t^{-1}$ for $t \in T$.

Lemma 6. *Let G be a connected reductive complex algebraic group and $K \subset G$ a maximal compact subgroup. Any Weyl involution θ of K extends uniquely to a Weyl involution of G.*

Proof. By Theorem 5.2.11 in [OV] any differentiable automorphism $K \to K$ extends uniquely to a polynomial automorphism $G \to G$. Let $\theta : K \to K$ be a Weyl involution and $T \subset K$ a maximal torus of K on which $\theta(t) = t^{-1}$. Now, the complexification $T^{\mathbb{C}}$ of T is a maximal torus of G. The extension of θ to G, which we again denote by θ, is a Weyl involution of G because $\theta(t) = t^{-1}$ for all $t \in T^{\mathbb{C}}$. □

An algebraic subgroup $H \subset G$ is called *spherical* if G/H is a spherical variety, i.e., if a Borel subgroup of G acts on G/H with an open orbit. A reductive algebraic subgroup $H \subset G$ is called *adapted* if there exists a Weyl involution $\theta : G \to G$ such that $\theta(H) = H$ and $\theta|_{H^\circ}$ is a Weyl involution of the connected component H°. A similar definition is used for compact subgroups of connected compact Lie groups.

Proposition 2. *Any spherical reductive subgroup $H \subset G$ is adapted.*

Proof. See [AV], Proposition 5.10. □

Proposition 3. *Let $H \subset G$ be an adapted algebraic subgroup, $K \subset G$ and $L \subset H$ maximal compact subgroups, and $L \subset K$. Then L is adapted in K.*

Proof. See [AV], Proposition 5.14. □

Theorem 3. *Let G be a connected reductive algebraic group and $H \subset G$ a reductive spherical subgroup. Then there exist a Weyl involution $\theta : G \to G$ and a Cartan involution $\tau : G \to G$ such that $\theta\tau = \tau\theta$, $\theta(H) = H$ and $\tau(H) = H$.*

Proof. Let L be a maximal compact subgroup of H and K a maximal compact subgroup of G that contains L. Then K is the fixed point subgroup G^τ of some Cartan involution τ. Since H is adapted in G, there is a Weyl involution $\theta : K \to K$ such that $\theta(L) = L$. For any $k \in K$, we have $\theta\tau(k) = \theta(k) = \tau\theta(k)$ by the definition of τ. Denote again by $\theta : G \to G$ the unique extension to G of the given Weyl involution of K. Since G is connected and the relation $\theta\tau(g) = \tau\theta(g)$ holds on K, it also holds on G. □

Theorem 4. *Let $X = G/H$ be an affine homogeneous space of a connected reductive algebraic group G. Let K be a maximal compact subgroup of G. Then X is spherical if and only (FT) is satisfied for the action of K on X. Moreover, if X is spherical one can choose μ in (FT) to be θ-equivariant, where θ is a Weyl involution of K.*

Proof. (FT) implies that $\mathcal{O}(X)$ is multiplicity free or, equivalently, that X is spherical. Conversely, assume that $X = G/H$ is a spherical variety. Since X is affine, H is a reductive subgroup by Matsushima-Onishchik theorem. Define θ and τ as in Theorem 3 and put $\mu(g \cdot H) = \theta\tau(g) \cdot H$. The map $\mu : X \to X$ is well defined because $\theta\tau(H) = H$. The lift of μ to G is an antiholomorphic involutive automorphism, so it is obvious that μ is an antiholomorphic involution of X. Since $\theta\tau = \tau\theta$, it follows that $\theta(K) = K$. Therefore, for any $x = gH \in X$ one has

$$\mu(kx) = \theta\tau(kg) \cdot H = \theta(k)\theta\tau(g) \cdot H = \theta(k)\mu(x)$$

for all $k \in K$. From Theorem 2 it follows that $\mu(x) \in Kx$ for all $x \in X$. □

6. Non-spherical spaces: an example

We keep the notation of the previous section. For a spherical affine homogeneous space $X = G/H$, we constructed a θ-equivariant antiholomorphic involution μ. In this section we want to show that the sphericity assumption is essential.

Lemma 7. *Let X be an irreducible reduced complex space with a holomorphic action of G. Let θ be any algebraic automorphism of G preserving a maximal compact subgroup $K \subset G$. Denote by τ the Cartan involution with fixed point subgroup K. If μ is an antiholomorphic involution of X satisfying $\mu(kx) = \theta(k)\mu(x)$ for all $x \in X,\ k \in K$, then one has $\mu(gx) = \theta\tau(g)\mu(x)$ for all $x \in X,\ g \in G$.*

Proof. For every fixed $x \in X$ consider two antiholomorphic maps $\varphi_x : G \to X$ and $\psi_x : G \to X$, defined by $\varphi_x(g) = \mu(gx)$ and $\psi_x(g) = \theta\tau(g)\mu(x)$. Since the required identity holds for $g \in K$, the maps φ_x and ψ_x coincide on K. But K is a maximal totally real submanifold in G, so φ_x and ψ_x must coincide on G. □

Lemma 8. *Let $X = G/H$, where $H \subset G$ is an algebraic reductive subgroup, θ a Weyl involution of G preserving K, and $\mu : X \to X$ an antiholomorphic involution of X satisfying $\mu(kx) = \theta(k)\mu(x)$. Then H and $\theta(H)$ are conjugate by an inner automorphism of G.*

Proof. Assume first that $\tau(H) = H$. Let $x_0 = e \cdot H$ and $h \in H$. Then $\theta(h)\mu(x_0) = \theta(\tau(\tau(h)))\mu(x_0) = \mu(\tau(h)x_0) = \mu(x_0)$ by Lemma 7. It follows that $\theta(H)$ is the stabilizer of $\mu(x_0)$, so H and $\theta(H)$ are conjugate.

To remove the above assumption, take a maximal compact subgroup $L \subset H$ and a maximal compact subgroup $K_1 \subset G$, such that $L \subset K_1$. Then $K_1 = gKg^{-1}$ for some $g \in G$. The fixed point subgroup of the Cartan involution $\tau_1 := \mathrm{Ad}(g)\tau\mathrm{Ad}(g)^{-1}$ is exactly K_1, so τ_1 is the identity on L and, consequently, $\tau_1(H) = H$. Let $H_1 := g^{-1}Hg$, then $\tau(H_1) = (\mathrm{Ad}g)^{-1}\tau_1(\mathrm{Ad}g)(H_1) = (\mathrm{Ad}g)^{-1}\tau_1(H) = (\mathrm{Ad}g)^{-1}(H) = g^{-1}Hg = H_1$. Replacing H by H_1, we can apply the above argument. □

Proposition 4. *Let $G = \mathrm{SO}_{10}(\mathbb{C})$ and $H \subset G$ the adjoint group of $\mathrm{SO}_5(\mathbb{C})$. Let θ be a Weyl involution of the maximal compact subgroup $K = \mathrm{SO}_{10}(\mathbb{R})$. Then an antiholomorphic involution of $X = G/H$ cannot be θ-equivariant.*

Proof. Extend θ holomorphically to G. In view of Lemma 8 it suffices to show that $\theta(H)$ and H are not conjugate by an inner automorphism of G. Assume that $\theta(H) = g_0Hg_0^{-1}$. Then there is an automorphism $\phi : H \to H$, such that $\theta(h) = g_0\phi(h)g_0^{-1}$ for $h \in H$. All automorphisms of H are inner, so $\phi(h) = h_0hh_0^{-1}$ for some $h_0 \in H$. Therefore $\theta(h) = g_1hg_1^{-1}$ for all $h \in H$, where $g_1 = g_0h_0$. Define

an automorphism of $\alpha : G \to G$ by $\alpha := (\mathrm{Ad}(g_1))^{-1} \cdot \theta$ and remark that $H \subset G^\alpha$, where G^α is the fixed point subgroup of α.

Recall that E.B.Dynkin classified maximal subgroups of classical groups in [D]. Since B_2 does not occur in his Table 1, every irreducible representation of B_2 defines a maximal subgroup by Theorem 1.5 in [D]. In particular, H is a maximal connected subgroup in G. Therefore, either (i) H is the connected component of G^α or (ii) $\alpha = \mathrm{Id}$. Now, (ii) implies that $\mathrm{Ad}(g_1) = \theta$, thus θ is an inner automorphism of G, which is not the case. So we are left with (i). Applying the same argument to $\beta = \alpha^2$, we see that either (i1) H is the connected component of G^β or (i2) $\beta = \mathrm{Id}$. Since β is certainly an inner automorphism, (i1) would imply that H is the centralizer of an element of G. However, all centralizers have even codimension in G and $\mathrm{codim}(H) = 35$. On the other hand, if (i2) were true then H would be a symmetric subgroup in G. The list of symmetric spaces shows that this is not the case. The contradiction just obtained completes the proof. □

References

[A] D.N.Akhiezer, Lie group actions in complex analysis, Vieweg, Braunschweig - Wiesbaden, 1995.

[AH] D. N. Akhiezer, P. Heinzner, Spherical Stein spaces, Manuscripta Math. 114, 327-334 (2004)

[AV] D. N. Akhiezer, E. B. Vinberg, Weakly symmetric spaces and spherical varieties, Transform. Groups, Vol. 4, No. 1, 1999, 3-24

[D] E. B. Dynkin, Maximal subgroups of the classical groups, Amer. Math. Soc. Transl., Ser. 2, vol. 6 (1957), 245 - 378

[FT] J. Faraut, E. G. F. Thomas, Invariant Hilbert spaces of holomorphic functions, J. Lie Theory, Vol. 9 (1999), 383-402

[H-C] Harish-Chandra, Discrete series for semisimple Lie groups II, Acta Math. 116 (1966), 1-111

[OV] A. L. Onishchik, E. B. Vinberg, Lie groups and algebraic groups, Springer 1990

Institute for Information Transmission Problems, B.Karetny 19, 101447 Moscow, Russia

E-mail address: akhiezer@mccme.ru

Ruhr-Universität Bochum, Fakultät für Mathematik, Universitätsstrasse 150, 44780 Bochum, Germany

E-mail address: annett.puettmann@rub.de